Iterative Decoding of Turbo Product Codes over PR-equalized Lorentzian Channels with Colored Noise

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Abstract—**Following the trend of turbo codes and low density parity check (LDPC) codes, single-parity turbo product codes (TPC/SPC) are being seriously considered for application in future high-density recording systems. Recent work on TPC/SPC codes has focused on ideal partial response channels with additive white Gaussian noise. This work extends the investigation to a more realistic equalized Lorentzian channel model where imperfect channel shaping, colored noise and recording density effect are taken into consideration. The effect of precoding is discussed and the interleaving gain is quantified. Simulation results of the turbo decoding system with both channel models are presented. A comprehensive evaluation is conducted, including BER performance, code rate selection, equalization targets and error statistics, which demonstrate TPC/SPC codes to be a promising candidate for future high-density recording systems.**

I. INTRODUCTION

The increasing demand for vast, inexpensive and reliable data storage to satisfy the explosive growth of digitally stored information has resulted in vigorous research for ever increasing recording densities. Toward satisfying this demand, in addition to technology advances in recording heads, media, and servo control, to mention a few, signal processing and coding play an important role. From the perspective of information technology, a digital magnetic recording channel can be modeled as a noisy, dispersive communication channel with colored noise where many of the advanced techniques in signal processing and telecommunication theory can be used.

Recently, in the wake of turbo codes, low density parity check (LDPC) codes, turbo product codes (TPC) and related structures, concatenated schemes with iterative approaches are being seriously considered for use in future data storage systems, in the hope for further enhanced recording densities. Turbo codes have been under intensive investigation on magnetic recording channel models like ideal partial response (PR) channels and more realistic Lorentzian channels [1]-[3]. Performance gains of 3 to 6 dB over uncoded partial response maximum likelihood (PRML) systems are observed under various conditions. Then, a simplified serial concatenated structure involving only a single convolutional code of moderate constraint length is shown to yield similar, or in some cases even better, performance [3]-[5] (call it serial turbo system). However the improved performance of parallel/serial turbo systems comes at a price of increased complexity, because the convolutional codes require MAP (maximum *a posteriori* probability) decoding, whose trellis usually involves many states and is expensive to implement.

Soon afterwards, low density parity check codes, which provide comparable performance to turbo codes in AWGN channels at much smaller complexity, caught the attention of the data storage community [6]-[9]. The lack of convincing evidence that LDPC codes do not have an error floor

above 10^{-15} still necessitates the conventional use of an outmost Reed Solomon error correction codes (RS-ECC) to guard against the residual errors. Unfortunately, although capable of providing impressive coding gains, the error statistics of LDPC codes, and in particular the presence of a large number of errors within a single block, may easily exceed the capacity of the outer RS-ECC code [8] [9] . As a consequence, very little additional gains can be provided after the RS-ECC code, resulting in unacceptably high block failure rates.

In this work we focus on turbo product codes [10] [11], and in particular single parity check turbo product codes (TPC/SPC) for PR-equalized Lorentzian channels. In [8] and [12], an interleaved serial concatenated structure where the outer code is a TPC/SPC code and the inner code is the precoded ideal PR channel is studied. With similar (in fact a little less) complexity to LDPC codes, TPC/SPC codes are shown to provide around 5 dB gain when properly precoded, indicating a performance essentially the same as LDPC codes. Further, bit/byte error statistics are investigated which reveal a harmonious compatibility with the out-most RS-ECC code. However the research so far assumes ideal PR channels with independent and identically distributed (i.i.d.) Gaussian noise, which is inadequate to characterize the actual recording systems where channel shaping is imperfect and noise is correlated. This paper aims to extend the investigation of the above serial architecture by looking at a more realistic PR-equalized Lorentzian channel model with colored noise. This more realistic channel will lend a more credible evaluation of TPC/SPC codes for its potential in magnetic recording channels.

The paper is organized as follows. Section 2.1 presents a brief introduction to TPC/SPC codes. Section 2.2 presents the system model of both channels with focus on signal processing and iterative decoding techniques involved. Section 2.3 discusses precoding and interleaving gain. Section 3 offers a comprehensive performance evaluation and discussion. Summary and conclusions are given in Section 4.

II. TPC/SPC CODES OVER PR CHANNELS

A. Introduction to TPC/SPC Codes

Turbo product codes (TPC) [10] [11], also known as block turbo codes (BTC), are formed by concatenation of codewords from linear block codes in a multi-dimensional manner. Here, "turbo" refers to its iterative decoding approach where the overall soft-in soft-out (SISO) module operates by iterating soft information among the SISO sub-modules of its component codes (See Fig. 1(b)). "Product" refers to the fact that the code parameters of a TPC code are the product of those of its component codes. In other words, an m-dimensional TPC code, $\mathcal{C}$, constructed of component codes $\mathcal{C}_i \sim (n_i, k_i, d_i, G_i)$, $i = 1, 2, \cdots, m$, has parameters

$(\prod_i n_i, \prod_i k_i, \prod_i d_i, \boxtimes_i G_i)$, where n, k, d, G denotes the codeword size, user data size, minimum distance and generator matrix, respectively, and $\boxtimes$ denotes the Kronecker product.

Although not required by definition, the component codes of a turbo product code are usually chosen to be systematic, leading to the overall product code also being systematic (Fig. 1(a)). Further all component codes are preferred to be the same, for the simplicity of implementation as well as the efficiency in code rate. For a given code rate, it can be readily shown (by treating the lengths of component codes as continuous and using Lagrange multipliers) that the smallest block size is reached when all component codes are of equal length. It has been recognized that very simple (almost useless) component codes can form an overall powerful TPC code. Particularly of interest to the data storage systems is the type of turbo product codes formed from single parity check codes (denoted TPC/SPC) which has the intrinsic advantage of high rates, soft decodability, as well as linear encoding and decoding complexity [12] [10]. Further, both the encoding and decoding are highly parallelizable, a feature very attractive for hardware implementation.

B. System Model and Iterative Decoding

A transition sequence in magnetic recording systems are represented by non-return-to-zero inverted (NRZI) waveform modulation, where a bit "1" is signified by a transition and "0" no transition. The response of the head to a transition in magnetization along the track is modeled as a step function or transition response with a Lorentzian pulse $s(t)$:

$$s(t) = \frac{1}{1 + (2t/PW_{50})^2}, \tag{1}$$

where PW_{50} is the width of the pulse at 50% of its peak value. Since $s(t)$ is a response to NRZI dibit, the continuous time channel response is characterized by the dipulse:

$$h(t) = \frac{1}{2}[s(t) - s(t - T)] \tag{2}$$

where T is the channel bit duration.

The dipulse $h(t)$ has a Fourier transform $H(v)$ with a spectral null at $v = 0$ and when normalized linear densities $D_n = PW_{50}/T$ is greater than 2, most of its spectral energy is within the frequency band $[-1/2T, 1/2T]$. This DC-free spectrum and strong high-frequency attenuation validate the characterization of the channel in its discrete time domain by sampling at every T seconds, ie: $h_k = h(kT)$.

We consider electronic noise caused by the head circuitry in our system (denoted by n_e in Fig. 2), which is modeled as white Gaussian noise with uniform two-sided spectral density $N_0/2$. The read-back data is therefore a sampled sequence from a linear dispersive channel with additive white Gaussian noise:

$$r(t) = \left(\sum_i s_i \cdot p_s(t - iT) \right) * h(t) + n_e(t), \tag{3}$$

where s_i denotes the binary input, $p_s(t)$ the write current pulse, $n_e(t)$ the electronic noise, and $*$ denotes convolution operation. After low-pass filtering and sampling, a digital linear equalizer, which is a finite impulse response (FIR) filter, is employed to shape the channel response to a desired target of short duration and with amplitude-frequency characteristics closely matched to those of the channel. Since the noise is added prior to sampling, the front-end filtering and equalization will color the noise samples seen by the detector. In PRML systems, the equalized samples are handed to a Viterbi detector to decode the written user bits. Reasonable PR targets for equalization take the form of $H(D) = (1 - D)P(D)$, where $P(D) = 1 + p_1 D + p_2 D^2 + \cdots + p_L D^L$. Some popular targets are the partial response class IV family of the form $H(D) = (1 - D)(1 - D)^L$, where $L = 1$ called PR4, $L = 2$ EPR4, $L = 3$ E^{2}PR4 and etc. For the ideal PR channel model, the channel impulse is modeled as a perfectly equalized partial response polynomial with additive white Gaussian noise as illustrated in the block diagram in Fig. 2.

When TPC/SPC codes are used in conjunction with the ideal PR channels or PR-equalized Lorentzian channels, SISO decoders for channel and the code are employed for iterative detection. Once the noisy observations are received, the channel MAP decoder starts processing them (no *a prior* information at this point yet) and producing output soft probabilities known as the *extrinsic* information which is deinterleaved and passed as *a prior* information into the outer TPC/SPC decoder. The extrinsic output from TPC/SPC decoder in return provides *a prior* information to the inner channel decoder. The random interleaver in-between works to break up the correlation among the data sequence, to avoid bursty error patterns and, in conjunction with the precoder, to offer a random permutation of the TPC/SPC codewords so that low-weight error events will hopefully map out to high-weight ones. This *spectrum thinning* phenomenon is what leads to the *interleaving gain*. After a certain number of message iterations, the overall soft output is obtained by combining the extrinsic information from both decoders. Fig. 3 illustrates the message flow procedure. An explicit description of the decoding steps can be found in [12].

C. Precoding and Interleaving Gain

Precoding makes a general inter-symbol interference (ISI) channel appear like a rate-1 recursive convolutional code to the outer code. As mentioned above, the recursiveness of the inner code and the randomness of the interleaver combinatively contribute to the interleaving gain. From Benedetto, Divsalar, McEliece *et al*'s results [13] [14], at large E_b/N_o, the word error rate of a serial concatenation with recursive inner code is upbounded by:

$$P_w^{UB} \propto O(N^{-\lfloor \frac{d_m^o - 1}{2} \rfloor}), \quad N \to \infty, \tag{4}$$

where N is the interleaver size, and d_m^o is the the minimum distance of the outer code. In other words, the interleaving gain is at least reversely proportional to the interleaver size provided that the outer code has a minimum distance of at least 3. Since the minimum distance of TPC/SPC codes is 4, it is tempting to directly apply the above result. However, one has to be careful in interpreting it. In our system setting, q blocks of $(k + 1, k)^2$ TPC/SPC codewords are combined and jointly passed into the interleaver of size $N = q(k + 1)^2$. This

is important for two reasons. (1) It lends flexibility to code rate and block size. With $q = 1$, it is hard to find a TPC/SPC code that has both the proper block size and the proper code rate for use in magnetic recording systems ($\sim$4K bits/block, $R = 0.88 \sim 0.95$). (2) Furthermore, as shown in [12], the upper bound of the word error probability in TPC/SPC system is reversely proportional to q rather than N:

$$P_{w,TPC/SPC}^{UB} \propto O(q^{-1}), \quad N \to \infty. \tag{5}$$

Hence, unless several blocks of TPC/SPC codewords are combined and jointly interleaved, no interleaving gain will be obtained.

It has been shown in [15] that precoding incurs loss in the initial channel conditions, but the interleaving gain it provides will eventually lead to a performance better than that of the non-precoding case given moderate iteration numbers. Further, not only the error probability, but the convergence rate and the error floors may also vary (substantially) with different precoders [8]. It has been shown in theory [15] and through simulations [8] that the best precoder for PR IV family takes the form of $1/(1 \oplus D^2)$, where $\oplus$ denotes modulo 2 addition. Since $1/(1 \oplus D^2)$ has memory of 2, smaller than those of the channels (in the PR IV family), it adds no additional complexity to the channel MAP decoder.

III. NUMERICAL RESULTS

A. Simulation Specifications

The codes we consider in this paper have rate 0.89 and rate 0.94, respectively. Both have effective data block size of 4K bits, corresponding to a sector of 512 user bytes. The rate-0.89 code is formed by combining 16 codewords of a $(17, 16)^2$ 2-D TPC/SPC code, and the rate-0.94 codes is formed from 4 codewords of a $(33, 32)^2$ 2-D TPC/SPC code.

For a fair comparison of the various schemes at different rates and normalized linear densities D_n, adjustment of the normalized density for the rate loss is needed. In other words, the physical recording density for a rate-R code is $D_{user} = D_n/R$. Unless otherwise stated, performance curves presented employ no more than 5 iterations. In each iteration, the turbo equalization procedure starts by executing forward-backward process in the channel MAP decoder, followed by two rounds of bit-check updates in the outer TPC/SPC decoder, with (extrinsic) information in the form of log likelihood ratios (log-LLR) exchanged in between. We have found this schedule to yield the best performance with the least complexity. In the TPC/SPC concatenated architecture, a precoder in the form of $1/(1 \oplus D^2)$ is always adopted to make the channel appear recursive.

For the ideal PR channels with additive while Gaussian noise, we define the signal to noise ratio (SNR) as:

$$SNR = 10 \cdot \log_{10}\left(\frac{E_b}{N_o}\right) = 10 \cdot \log_{10}\left(\frac{E_s}{2R\sigma^2}\right), \tag{6}$$

where E_s is the symbol energy, $\sigma^2 = N_y 0/2$ is the noise variance, and R is the code rate.

For equalized Lorentzian channel model, we define:

$$SNR = 10 \cdot log_{10}\left(\frac{\bar{S}}{\bar{N}}\right), \tag{7}$$

where $\bar{S}$ and $\bar{N}$ are the mean square of the signal value and the noise value measured at the input to the equalizer, respectively.

B. Simulation Results

Performance over ideal channel model — Fig. 4 shows the performance of TPC/SPC codes over ideal PR channel model. For comparison purpose, performance of LDPC codes with similar rates as well as uncoded PRML systems are plotted in the same graph. The comparing LDPC codes are regular with uniform column weight of 3, user data block size of 4K bits, and rate $8/9 = 0.89$ and $16/17 = 0.94$ respectively. As can been, TPC/SPC and LDPC codes have almost identical performance, both achieving gains of around 5 dB over uncoded PRML systems at BER of 10^{-5}.

Performance over Lorentzian channel model — To evaluate the performance of TPC/SPC codes over Lorentzian channels and to examine the trade-off between code rate loss and coding gain, we tested rate-0.89 and rate-0.94 TPC/SPC codes (user data block size 4K bits) at several normalized densities over a EPR4-equalized Lorentzian channel. For a given normalized density in a given channel, lower rate codes provide more error correction capabilities but encounter more ISI, whereas higher rate codes are intrinsically weaker but encounter less ISI. Hence a balance in choosing code rate for different areal densities must be maintained for a best hit. The curves in Fig. 5 clearly indicate that lower rate codes work more comparably with low densities, while higher rate codes are better for high densities. Performance of uncoded PRML systems at normalized density of $D_n = 3.0, 3.25, 3.5$, is also presented for comparison. Gains of 6 to 7 dB are obtained by TPC/SPC systems, which is comparable to, or slightly better than, the serial turbo systems studied in [3], yet TPC/SPC systems have much lesser decoding complexity than serial turbo systems.

PR targets at different densities — For a given normalized areal density, the bit error rate (BER) performance of the code is contingent to the equalized PR targets among other factors. In this work, we investigated the performance of a rate-0.94 TPC/SPC code over different PR targets at normalized density $D_n = 3.5$. Fig. 6 shows that E^2PR4 is a better target than EPR4 for TPC/SPC codes at $D_n = 3.5$, and that ME^2PR4 ($H(D) = 5 + 4D - 3D^2 - 4D^3 - 2D^4$) seems to offer even larger gains (additional 1.2 dB gain over E^2PR4 at $D_n = 3.5$). But E^2PR4 and ME^2PR4 targets require a 16 state trellis for channel decoding which doubles the complexity of EPR4 channel decoding.

Error statistics — Error statistics serve as a crucial criterion for it directly affects the effective functioning of the out-most RS-ECC code, which in turn translates to the overall block failure rate. While LDPC codes typically encounter a few dozens of errors in one block [9] [8], such error bursts are very rare in TPC/SPC codes. We attribute this to the fact that TPC/SPC codes have quite small minimum distance. It is perceived that when error occurs, the decoder is mostly likely to decode it to its nearest neighbor which fortunately is not different in too many bit positions. In [8] and [12], the error statistics of PR4 and EPR4 ideal channels are investigated, and shown that the maximum number of bit/byte errors observed within a block, M_e, does not exceed 20 at BER of around 10^{-5} or SER (symbol error rate) of around 10^{-4} over

100,000 blocks tested. To facilitate the understanding of the effect of colored noise and imperfect channel shaping, error statistics are examined for more realistic Lorentzian channels. As shown in Fig. 7, the same good news seem to prevail. After 3 iterations, no blocks containing more than 14 symbols[1] over 10,000 blocks transmitted. Although not shown, we also examined EPR4- and E^2PR4-equalized Lorentzian channels. Similar phenomena are observed. Although reliable statistics require the experiments of millions of blocks, which is infeasible for software simulation, the preliminary results we provide in this work indicate that (1) TPC/SPC codes are quite insensitive to colored noise (due to the random interleaver) and (2) TPC/SPC codes tend to work in good harmony with the out-most RS-ECC codes such that the overall capacity is maximized.

IV. CONCLUSION

This paper conducts a comprehensive performance analysis of TPC/SPC codes. Both the early work of ideal partial response channels and a more realistic system model of Lorentzian channels with colored noise are discussed, with highlights on the signal processing and coding techniques involved. By combining several blocks of TPC/SPC codewords before interleaving and by properly precoding the PR channels, interleaving gain is achieved. Gains of around $5 - 7$ dB at BER of 10^{-5} are obtained over uncoded PRML on ideal PR channels and PR-equalized Lorentzian channels, revealing performance comparable to those of LDPC and serial turbo codes. The gains are more substantial when the Lorentzian channel is equalized with a more proper target such as E^2PR4 and ME^2PR4 at high densities (at the cost of increased complexity). Further, error statistics are studied indicating that TPC/SPC codes are quite robust with colored noise and inaccurate channel shaping, and that they are in better compatibility with the out-most RS-ECC codes than LDPC codes, which suggests its feasibility in future magnetic recording systems.

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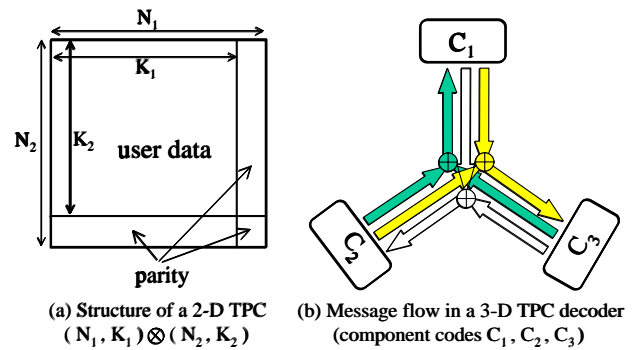
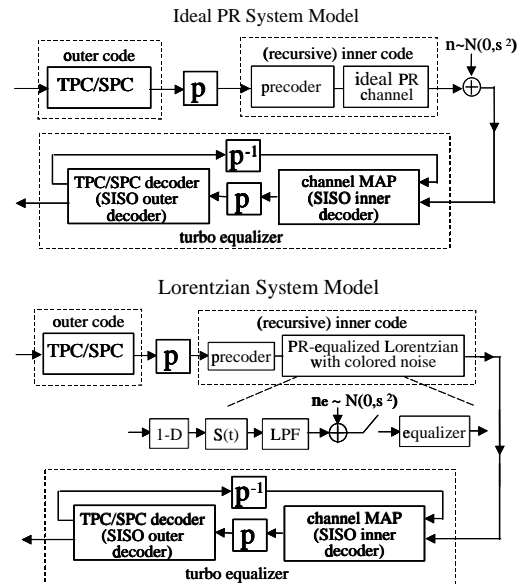Fig. 1. Structure of TPC/SPC codes and iterative decoding

Fig. 2. System model.

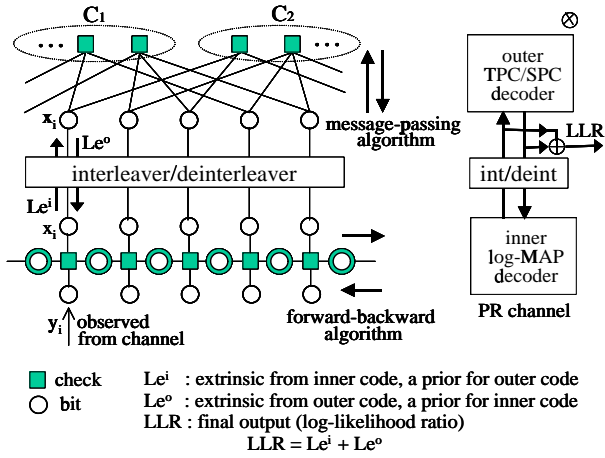[1] Here a symbol is equivalent to a byte, which contains 8 consecutive bits.

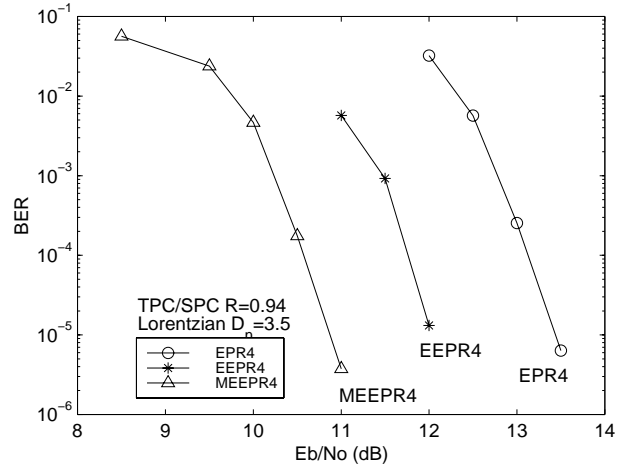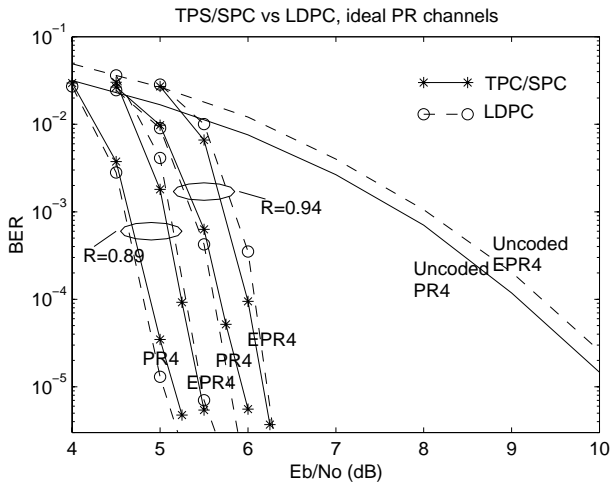Fig. 3. Illustration of message flow

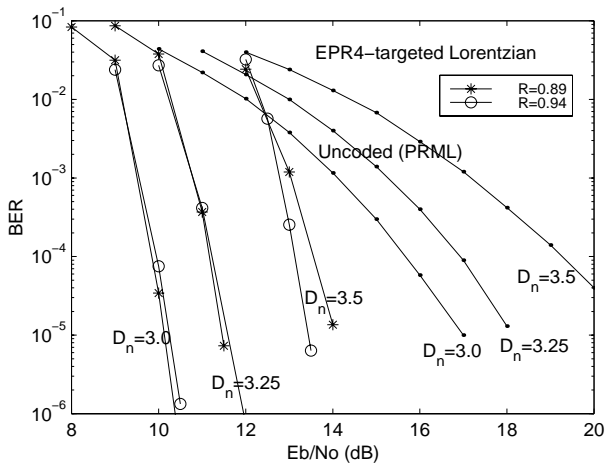
Fig. 4. Performance of TPC/SPC codes over ideal channels

Fig. 5. Performance of TPC/SPC codes over EPR4-equalized Lorentzian channels at different code rates and different densities (rate $R = 0.89, 0.94$, normalized density $D_n = 3.0, 3.25, 3.5$)
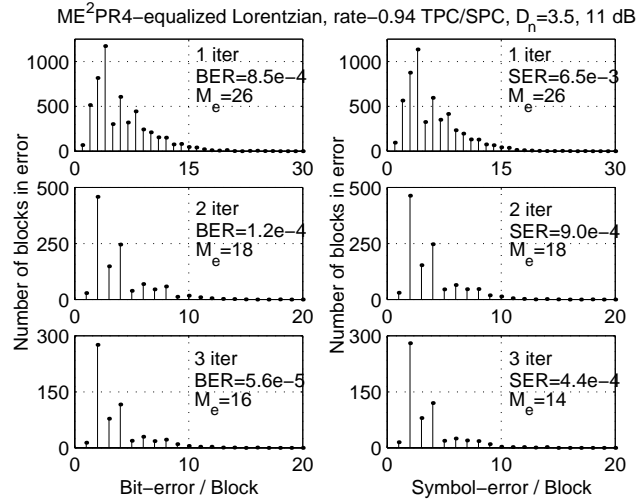

Fig. 6. Effect of PR targets (PR targets: EPR4, E^2PR4, ME^2PR4, Normalized Density $D_n = 3.5$, rate $R = 0.94$)

Fig. 7. Bit/Byte error statistics of TPC/SPC codes (ME^2PR4-equalized Lorentzian, E_b/N_o=11dB, $D_n = 3.5$, $R = 0.94$, iterations=1, 2, 3, "BER"–bit error rate, "SER"–symbol error rate, "M_e"–maximum number of errors observed in a block)