Product Accumulate Codes: Properties and Performance

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Abstract — **A new class of codes, named *product accumulate* codes, which are the concatenation of an outer product code and an inner rate-1 differential encoder (or accumulator) is proposed. We show that these codes can perform within a few tenths of a dB from the Shannon limit for rates $\geq 1/2$. For practical block lengths, these codes provide similar performance to turbo codes but with significantly lower decoding complexity.**

I. Introduction

We propose a novel class of provably "good" codes [1] which are referred to as *product accumulate* (PA) codes. The proposed codes are shown to possess many desirable properties, including close-to-capacity performance, low decoding complexity, regular structure and easy rate adaptivity uniformly for all rates $R \geq 1/2$.

II. Encoder and Decoder Structure

The proposed structure is shown in Fig. 1. An input block of $K = Pt$ data bits is encoded using a parallel concatenated code whose component codes are $(t+1, t)$ single parity check (SPC) codes. The data bits and the parity bits from the parity check codes are further interleaved and encoded by a rate-1 inner code resulting in the overall rate being $t/t+2$. Since the outer code can be considered as a turbo product code with single parity check component codes (TPC/SPC) with a random interleaver instead of the conventional block interleaver, we call these codes as product accumulate codes.

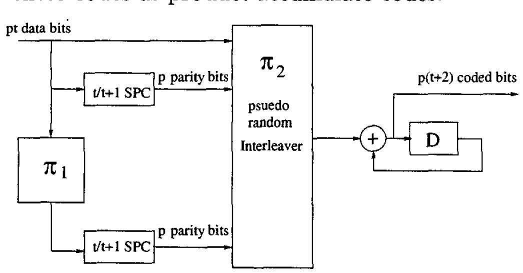

Figure 1: System model of PA codes

The turbo principle is used to iteratively decode the serially concatenated system, where soft extrinsic information in log-likelihood ratio (log-LLR) form is exchanged between the inner and outer decoders. The decoding of the outer TPC/SPC code is done using a message-passing algorithm similar to that of LDPC codes. The inner rate-1 convolutional code is typically decoded using a 2-state Bahl, Cocke, Jelinek, Raviv (BCJR) algorithm. However, a more computationally efficient approach is to use message-passing decoding directly on the Tanner graph of the entire product accumulate code including the inner code [2].

III. Properties of PA codes

It is straightforward to see that PA codes are linear time encodable and they have a simple encoding algorithm. In this section, we also analyze the performance of PA codes with maximum likelihood (ML) decoding and under iterative decoding.

<u>Performance under ML Decoding</u>: We first show that under maximum likelihood decoding, the probability of word error is proportional to $P^{\ 1}$ for large E_b/N_0, where P is the number of TPC/SPC codewords concatenated before interleaving.

(A) Interleaving Gain

From the results of Benedetto *et al* [3] and Divsalar, Jin and McEliece [4], we know that for a general serial concatenated system with recursive inner code, there exists a threshold γ such that for any $E_b/N_0 \geq \gamma$, the asymptotic word error rate is upper bounded by:

$$P_w^{UB} = O\left(N^{\lfloor \ \frac{d_m^o \ 1}{2} \rfloor}\right), \tag{1}$$

where d_m^o is the minimum distance of the outer code and N is the interleaver size. The result in (1) indicates that if the minimum distance of the outer code is at least 3, then an interleaving gain can be obtained. However, the outer codewords of PA codes (with random interleavers) have minimum distance of only 2. Below we show that although the minimum distance of the outer codewords is only 2 over the ensemble of interleavers, an interleaving gain still exists for PA codes with random interleavers. Since from (1) outer codewords of weight 3 or more will lead to an interleaver gain, we focus the investigation on weight-2 outer codewords only and show that the number of such codeword vanishes as P increases. The all-zero sequence is used as the reference since the code is linear.

It is convenient to employ the uniform interleaver which represents the average behavior of the ensemble

[1] A "good" code is defined as a code for which there exists a threshold above which an arbitrarily low error rate can be achieved as block size goes to infinity [1].

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of codes. Let $A^{(j)}_{w,h}, j = 1, 2$, denote the *input output weight enumerator* (IOWE) of the j_{th} SPC branch code (parallelly concatenated in the outer code). The IOWE of the outer code, $A^{o}_{w,h}$, averaged over the code ensemble is given as:

$$A^{o}_{w,h} = \sum_{h_1} \frac{A^{(1)}_{w,h_1} A^{(2)}_{w,h\ \ h_1}}{\binom{K}{w}}, \tag{2}$$

where $K = Pt$ is the input sequence length.

Define the *input output weight transfer probability* (IOWTP) of the j_{th} branch code, $P^{(j)}_{w,h}$, as the probability that a particular input sequence of weight w is mapped to an output sequence of weight h $P^{(j)}_{w,h} = \frac{A^{(j)}_{w,h}}{\binom{K}{w}}$, $j = 1, 2$. Substituting this in (2), we get:

$$A^{o}_{w,h} = \sum_{h_1} A^{(1)}_{w,h_1} P^{(2)}_{w,h\ \ h_1}. \tag{3}$$

For each branch where P $(t+1, t)$ SPC codewords are combined, the IOWE function is given as (assuming even parity check):$A^{SPC}(w, h) = \left(\sum_{j=0}^{t} \binom{t}{j} w^j h^{2\lceil j/2 \rceil}\right)^P$, where the coefficient of the term $w^u h^v$ denotes the number of codewords with input weight u and output weight v. Using the above equation, we can compute the IOWEs of the first SPC branch code, denoted as $A^{(1)}_{u,v}$ $(= A^{SPC}_{u,v})$. For the second branch of the SPC code, since only parity bits are transmitted, $A^{(2)}_{u,v} = A^{(1)}_{u,v+u}$. With a little computation, it is easy to see that the number of weight-2 outer codewords is given by:

$$A^{o}_{h=2} = \sum_{w} A^{o}_{w,h=2} = P\binom{t}{2}\left(\frac{\binom{P\ \ t}{2}}{\binom{Pt}{2}}\right) = O(t^2) \tag{4}$$

where the last equation assumes a large P (i.e. large block size). Equation (4) shows that the number of weight-2 outer codewords is a function of a single parameter, t, which is related only to the rate of SPC codes and not the block length. Now considering the serial concatenation of the outer codewords with the inner $1/(1+D)$ code, the overall output weight enumerator (OWE), A^{PA}_{h}, is:

$$A^{PA}_{h} = \sum_{h'} A^{o}_{h'} \frac{A^{1/(1+D)}_{h',h}}{\binom{N}{h'}} = \sum_{h'} \sum_{w} A^{o}_{w,h'} \frac{A^{1/(1+D)}_{h',h}}{\binom{N}{h'}} \tag{5}$$

where $A^{o}_{h'}$ is the OWE of the outer code, and the IOWE of the $1/(1+D)$ code is given by [4]:

$$A^{1/(1+D)}_{w,h} = \binom{N\ \ h}{\lfloor w/2 \rfloor}\binom{h\ \ 1}{\lceil w/2 \rceil\ \ 1}. \tag{6}$$

In particular, the number of weight-s PA codewords produced by weight-2 outer codewords (for small-s), denoted as $A^{PA2}_{h=s}$, is:

$$A^{PA2}_{h=s} = \frac{(t\ \ 1)^2}{2} \frac{N\ \ s}{\binom{N}{2}} = O(tP\ \ ^1) \tag{7}$$

where $N = P(t+2)$ is the PA codeword length. This indicates that an interleaving gain results for PA codes.

(B) Upper bounds

To further shed insight into the asymptotic performance ($N \to \infty$) of PA codes under ML decoding, we compute thresholds for this class of codes based on the bounding technique recently proposed by Divsalar [5], which is tighter than the union bound for small E_b/N_o.

We first quote and summarize the main results of [5]. Define the *ensemble spectral shape* of a code, $\gamma(\delta)$, as the normalized weight distribution averaged over the code ensemble $\mathcal{C}_N$:

$$\gamma(\delta) \triangleq \lim_{N\to\infty} \frac{1}{N} \ln(A_{h=\lfloor \delta N \rfloor}), \quad 0 < \delta < 1, \tag{8}$$

where N is the code length, A_h is the (average) output weight enumerator of the code. It can be shown that the probability of word error can be upper bounded by $P_w(e) \leq \sum_h e^{\ \ NE(E_b/N_0,h)}$ [5]. The threshold $\mathbf{C}^*_{ML}$ is defined as the minimum E_b/N_0 such that $E(E_b/N_0, h)$ is positive for all h and, hence, for all $E_b/N_0 \geq \mathbf{C}^*_{ML}$, $P_w(e) \to 0$ as $N \to \infty$. The threshold can be computed as [5]:

$$\mathbf{C}^*_{ML} = \frac{1}{R} \max_{0<\delta\leq(1\ \ R)} \frac{1\ \ \delta}{2\delta}\left(1\ \ \ e^{\ \ 2\gamma(\delta)}\right) \tag{9}$$

where R is the code rate.

There is no simple closed form expression for the ensemble spectral shape of PA codes. However, the spectral shape can be computed to a good accuracy numerically since the component codes of the concatenation are single parity check codes. Specifically, using (3), (5) and (8) we can compute the spectral shape of PA codes, which is a function of N, P, t. We approximate the ensemble spectral shape by choosing a large N. Whenever possible, input output weight transfer probability, $P_{w,h}$, should be used instead of input output weight enumerator, $A_{w,h}$, to eliminate numerical overflow. The bounds for PA codes are computed and plotted in Fig. 2 (the simple bound and the union bound are shown). Also shown are the bounds for random codes and the Shannon limit. It can be seen that (1) the simple bounds of PA codes are very close to bounds computed for random codes, indicating that PA codes have good distance spectrum (2) the higher the rate, the closer the bound to that of random codes, indicating that GPA codes are likely more advantageous at high rates than low rates (as opposed to repeat accumulate codes).

<u>Performance under Iterative Decoding</u>: In this section, we compute thresholds for PA codes using density evolution (DE) [6]. Assuming the all-zeros transmitted sequence, the pdfs of the messages being passed within the inner code, outer code and between the two can be evolved since all the component codes are simple parity checks in this case. The pdfs are computed numerically and, hence, no assumptions are made on the pdfs of the extrinsic information. The outer code (alone) can also be

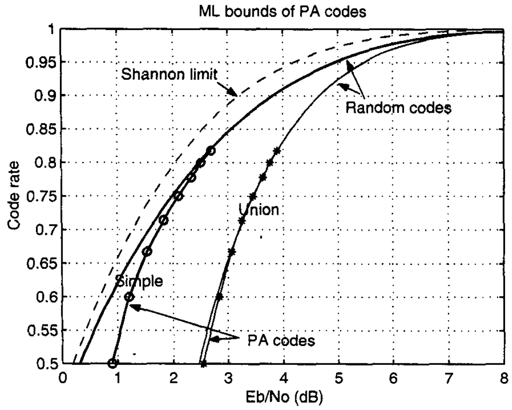

Figure 2: The union bound and the simple bound of product accumulate codes

considered as a special case of LDPC codes whose parity check matrix has $2P$ rows with uniform row weight of $(t+1)$, and $p(t+2)$ columns with $\frac{t}{t+2}$ percent of the columns having weight 2 and the rest weight 1. However, for more efficient convergence, we could make use of the fact that the checks in the outer code can be divided into two groups (corresponding to the upper and lower branch, respectively) such that the corresponding sub-graph (Tanner graph) of each group is cycle-free. It thus leads to a serial message-passing mode where each group of checks take turns to update (as opposed to the parallel update of all checks in LDPC codes). Similarly, a serial update is used for the inner code rather than a parallel update. The thresholds are computed using

$$\mathbf{C}^*_{iterative} = \inf_{SNR} \left\{ SNR : \lim_{k\to\infty} \lim_{N\to\infty} \int_{\infty}^{0} f_{L_o^{(k)}(x)} dx \to \infty \right\},$$

where $f_{L_o^{(k)}(x)}$ is the pdf of the messages (extrinsic information) evaluated at the output of the outer decoder, during the kth iteration between the inner and outer decoders.

To conserve space the results are not plotted in a figure. The difference between thresholds and the Shannon limit for binary input AWGN channel was seen to be 0.4931, 0.4370, 0.449, 0.4551, 0.4405, 0.4389, 0.4328, 0.4047, 0.3866 for rates R=0.5, 0.6, 0.667, 0.7143, 0.75, 0.8, 0.8333, 0.8889, 0.9231, respectively. Simulation results for long block lengths (64 Kbits for rate-1/2 and 16K for others) show that BER of $10^{\ 5}$ can be obtained at E_b/N_o within 0.2 dB from the computed thresholds.

IV. Simulation Results

Fig. 3 shows the performance of a rate-1/2 PA code of data block sizes 64K, 4K and 1K. It can be seen that the larger the block size the better the performance which clearly depicts the interleaving gain. For comparison purpose, the performance of a (2K,1K) turbo code with 16-state component codes and that of the recently reported irregular repeat accumulate (IRA) codes of the same parameters are also shown. As can be seen, (2K, 1K) PA codes perform as well as the turbo codes at BER of $10^{\ 5}$ with no error floors. From [2], we can see that the decoding complexity of rate-1/2 PA codes with 30 iterations is approximately 1/16 that of a 16-state turbo code with 8 iterations. It is also important to note that the complexity savings are higher as the rate increases, since the decoding complexity of punctured turbo codes does not reduce with increasing rate, whereas the decoding complexity of PA codes is inversely proportional to the rate. It should also be noted that the curve of PA codes is somewhat steeper than that of turbo codes or irregular repeat accumulate codes, and therefore may outperform them at lower BERs.

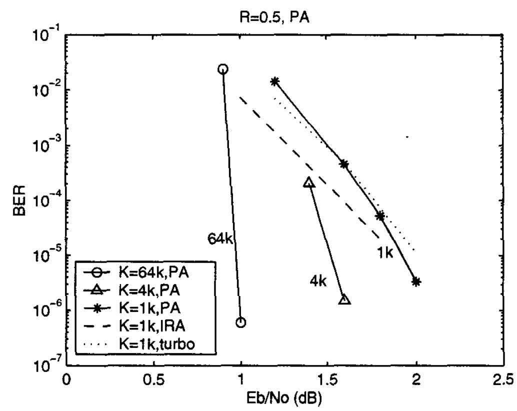

Figure 3: Performance of PA codes ate rate-1/2

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