BER Performance of Coded Free-Space Optical Links over Strong Turbulence Channels

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***Abstract*—Error control coding can be used over free-space optical (FSO) links to mitigate turbulence-induced fading. In this paper, we derive error performance bounds for coded FSO communication systems operating over atmospheric turbulence channels, which are modeled as *K* distribution under strong turbulence conditions. We derive an upper bound on the pairwise error probability (PEP) in closed form and then apply the transfer function technique in conjunction with the derived bound for PEP to obtain upper bounds on the bit error rate. Simulation results are further demonstrated to confirm the analytical results.**

***Keywords*—Atmospheric turbulence channel, free-space optical communication, pairwise error probability, error performance analysis.**

I. INTRODUCTION

Wireless optical communications, also known as free-space optical (FSO) communications, is a cost-effective and high bandwidth access technique and is receiving growing attention with recent commercialization successes [1]. With the potential high-data-rate capacity, low cost and particularly wide bandwidth on unregulated spectrum, FSO communications is an attractive solution for the "last mile" problem to bridge the gap between the end user and the fiber-optic infrastructure already in place. Its unique properties make it also appealing for a number of other applications, including metropolitan area network extensions, enterprise/local area network connectivity, fiber backup, back-haul for wireless cellular networks, redundant link and disaster recovery. In FSO communications, optical transceivers communicate directly through the air to form point-to-point line-of-sight links. One major impairment over FSO links is the atmospheric turbulence, which occurs as a result of the variations in the refractive index due to inhomogeneties in temperature and pressure fluctuations. The atmospheric turbulence results in fluctuations at the received signal, i.e. signal fading, also known as scintillation in optical communication terminology [2], severely degrading the link performance, particularly over link distances of 1 km or longer.

Error control coding as well as diversity techniques can be used over FSO links to improve the error rate performance [3-6]. In [5], Zhu and Kahn studied the performance of coded FSO links assuming a log-normal channel model for atmospheric turbulence. Specifically, they derived an approximate upper bound on the pairwise error probability (PEP) for a coded FSO communication system with intensity modulation/direct direction (IM/DD) and provided upper bounds on the bit error rate (BER) using the transfer function technique. Although lognormal distribution is the most widely used model for the probability density function (pdf) of the irradiance due to its simplicity, this pdf model is only applicable to weak turbulence conditions. As the strength of turbulence increases and multiple scattering effects must be taken into account, log-normal statistics exhibits large deviations compared to experimental data. One of the widely accepted models under strong turbulence regime is the *K distribution* [2]. This distribution was originally proposed to model non-Rayleigh sea echo [7], but it was also discovered that it provides good agreement with experimental data in a variety of experiments involving radiation scattered by strong turbulent media [8, 9]. It should be noted that *K* distribution was also proposed as a good approximation to Rayleigh-lognormal channels in the wireless RF communication literature [10] and used in the performance analysis [11]. However, one should be careful of the different underlying detection techniques in wireless optical and wireless RF systems: In a typical IM/DD FSO system the received current out of the optical detector is proportional to the square of the electromagnetic field and thus statistical models for atmospheric-induced turbulence (i.e. intensity fading) correspond to those applied to *power* in the coherent RF problem where the received current is proportional to the field and not its square. Therefore, the results in [11] can not be applied to performance analysis of FSO links in a straightforward manner.

In this paper, we will derive error performance bounds for coded FSO links operating over atmospheric channels, where the turbulence-induced fading is described by the *K-distribution*. The organization of the paper is as follows: In Section II, we review the K channel model under consideration. In Section III, an upper bound on PEP is derived and in section IV, we present numerical results to demonstrate the accuracy of the derived bound. Using transfer function technique in conjunction with the derived PEP expression, we also obtain bounds on the BER performance. Analytical results are further confirmed through Monte-Carlo simulation. Conclusions are presented in Section V.

II. THE K DISTRIBUTION

Most of the theoretical distributions proposed for the intensity fluctuations of an electromagnetic wave propagating through atmospheric turbulence are based on mathematical models which relate discreet scattering regions in the turbulent medium to the individual inhomogeneties in the electromagnetic wave. If the number of discrete scattering regions is sufficiently large, the radiation field of the electromagnetic wave is approximately Gaussian and therefore the irradiance statistics of the field are governed by the negative exponential distribution. This distribution model assumes very large number of scatterers and can be considered as a limiting case, thus can be used only for the *supersaturation* regime. The assumption of non-Gaussian radiation fields have led to various turbulence channel models, among which K distribution has been widely accepted as a succesful mathematical model for strong turbulence conditions, as also confirmed by experimental results [7, 8].

The K distribution [2] can be derived from a modulation process wherein the conditional pdf of irradiance is governed by the negative exponential distribution

$$f_{I|\mu}(I|\mu)=\frac{1}{\mu}\exp\left(-\frac{I}{\mu}\right),\ I>0 \tag{1}$$

with mean irradiance μ following the gamma distribution

$$f_{\mu}(\mu)=\frac{\alpha^{\alpha}\mu^{\alpha-1}}{\Gamma(\alpha)}\exp(-\alpha\mu),\ \mu>0\,. \tag{2}$$

Here $\Gamma(.)$ stands for the gamma function and α is a channel parameter related to the effective number of discrete scatterers. The unconditional distribution for the irradiance is then found as

$$\begin{aligned}f_{I}(I)&=\int_{0}^{\infty}f_{I|\mu}(I|\mu)f_{\mu}(\mu)dI\\&=\frac{2}{\Gamma(\alpha)}\alpha^{(\alpha+1)/2}I^{(\alpha-1)/2}K_{\alpha-1}\left(2\sqrt{\alpha I}\right),\ I>0\end{aligned} \tag{3}$$

where $K_a(.)$ is the modified Bessel function of the second kind of order a. In the limiting case of $\alpha\to\infty$, the gamma distribution approaches a delta function and the K distribution reduces to the negative exponential distribution.

III. DERIVATION OF PEP

We consider an IM/DD link using on-off keying (OOK). Following [5], we assume that the receiver signal-to-noise ratio is limited by shot noise caused by ambient light which is much stronger than the desired signal and/or by thermal noise. In this case, the noise can be modeled as additive white Gaussian noise (AWGN) with zero mean and variance $N_0/2$, independent of the on/off state of the received bit.

The pairwise error probability (PEP) represents the probability of choosing the coded bit sequence $\hat{\boldsymbol{X}}=(\hat{x}_1,\hat{x}_2,...,\hat{x}_M)$ when indeed $\boldsymbol{X}=(x_1,x_2,...,x_M)$ was transmitted. Here, we assume that the turbulence-induced fading remains constant over one bit interval and changes from one interval to another in an independent manner. Such an assumption can be justified by the use of perfect interleaving. Under the assumption of maximum likelihood soft decoding with perfect channel state information (CSI), the conditional PEP with respect to fading coefficients $I=(I_1,I_2,...,I_M)$ is given as [5]

$$P\left(\boldsymbol{X},\hat{\boldsymbol{X}}\middle|\boldsymbol{I}\right)=Q\left(\sqrt{\frac{\varepsilon\left(\boldsymbol{X},\hat{\boldsymbol{X}}\right)}{2N_0}}\right) \tag{4}$$

where $Q(.)$ is the Gaussian-Q function and $\varepsilon\left(\boldsymbol{X},\hat{\boldsymbol{X}}\right)$ is the energy difference between two codewords. Since OOK is used, the receiver would only receive signal light subjected to fading during on-state transmission. Therefore, under the assumption of OOK, (4) is simply given as

$$P\left(\boldsymbol{X},\hat{\boldsymbol{X}}\middle|\boldsymbol{I}\right)=Q\left(\sqrt{\frac{E_s}{2N_0}\sum_{k\in\Omega}I_k^2}\right) \tag{5}$$

where E_s is the total transmitted energy and Ω is the set of bit intervals' locations where $\boldsymbol{X}$ and $\hat{\boldsymbol{X}}$ differ from each other. Defining the signal-to-noise ratio as $\tau=E_s/N_0$ and using the alternative form for Gaussian-Q function, i.e. $Q(x)=(1/2\pi)\int_0^{\pi/2}\exp\left(-x^2/2\sin^2\theta\right)$ [12], we obtain

$$P(\boldsymbol{X},\hat{\boldsymbol{X}})=\frac{1}{\pi}\int_0^{\pi/2}\prod_{k\in\Omega}\exp\left(-\frac{E_s}{4N_0}\frac{I_k^2}{\sin^2\theta}\right)d\theta\,. \tag{6}$$

To obtain unconditional PEP, we need to take an expectation of (6) with respect to I_k. Using independency among fading coefficients I_k, we write

$$\begin{aligned}P(\boldsymbol{X},\hat{\boldsymbol{X}})&=\frac{1}{\pi}\int_0^{\pi/2}\prod_{k\in\Omega}\mathrm{E}_{I_k}\left[\exp\left(-\frac{\tau}{4}\frac{I_k^2}{\sin^2\theta}\right)\right]d\theta\\&=\frac{1}{\pi}\int_0^{\pi/2}\left[\int_0^{\infty}\exp\left(-\frac{\tau}{4}\frac{I^2}{\sin^2\theta}\right)f(I)dI\right]^{|\Omega|}d\theta\end{aligned} \tag{7}$$

where E(.) is the expectation operation and $|\Omega|$ is the cardinality of Ω, which also corresponds to the length of error event. Here, $f(I)$ is the pdf for the *K* channel given by (3). A direct use of (3) in (7) yields an expression which unfortunately does not have a closed form solution. One may resort to approximations as in [13], where the modified Bessel function is replaced by its infinite series representation. Instead here, we rewrite (7), exploiting the fact that the underlying distribution is a *conditional* negative exponential distribution with its mean μ following gamma distribution, i.e

$$P(\boldsymbol{X},\hat{\boldsymbol{X}})=\frac{1}{\pi}\int_0^{\pi/2}\left\{E_\mu\left\{E_{I|\mu}\left[\exp\left(-\frac{\tau}{4}\frac{I^2}{\sin^2\theta}\right)\right]\right\}\right\}^{|\Omega|}d\theta \tag{8}$$

The inner expectation in (8) gives

$$E_{I|\mu}\left[\exp\left(-\frac{\tau}{4}\frac{I^2}{\sin^2\theta}\right)\right]=\frac{1}{\mu}\int_0^{\infty}\exp\left(-\frac{\tau}{4}\frac{I^2}{\sin^2\theta}-\frac{I}{\mu}\right)dI\,. \tag{9}$$

Using the result from [14, p.113, Eq.2.33], i.e.

$$\begin{aligned}&\int\exp\left[-\left(az^2+2bz+c\right)\right]dz\,, \\&=\frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2-ac}{a}\right)\left[1-2Q\left(\sqrt{2a}z+\sqrt{\frac{2}{a}}b\right)\right],\ a\neq 0\end{aligned} \tag{10}$$

we obtain a closed form expression for (9) as

$$E_{I|\mu}\left[\exp\left(-\frac{\tau}{4}I^2\right)\right]=\frac{2}{\mu}\sqrt{\frac{\pi\sin^2\theta}{\tau}}\exp\left(\frac{\sin^2\theta}{\mu^2\tau}\right)Q\left(\sqrt{\frac{2\sin^2\theta}{\mu^2\tau}}\right) \tag{11}$$

Replacing $Q(\sqrt{z})\leq 0.5\exp(-z/2)$ in (11) and inserting the resulting expression in (8), we obtain

$$P(\boldsymbol{X},\hat{\boldsymbol{X}})\leq\frac{1}{\pi}\int_0^{\pi/2}\left[\frac{\alpha^\alpha}{\Gamma(\alpha)}\sqrt{\frac{\pi\sin^2\theta}{\tau}}\int_0^{\infty}\mu^{\alpha-2}\exp(-\alpha\mu)d\mu\right]^{|\Omega|}d\theta \tag{12}$$

where the inner integral can be easily solved with the help of [14, p. 364, eq. 3.381.4]

$$\int_0^{\infty}z^{\nu-1}\exp(-\lambda z)dz=\lambda^{-\nu}\Gamma(\nu),\ \ \mathrm{Re}(\lambda)>0\,,\ \mathrm{Re}(\nu)>0\,, \tag{13}$$

giving the final form for PEP as

$$P(\boldsymbol{X},\hat{\boldsymbol{X}})\leq\frac{1}{\pi}\int_0^{\pi/2}\left[\sqrt{\frac{\pi\sin^2\theta}{\tau}}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha)}\alpha\right]^{|\Omega|}d\theta\,. \tag{14}$$

It should be noted that inserting $\theta=\pi/2$ in the above PEP expression yields a Chernoff-type bound

$$P(\boldsymbol{X},\hat{\boldsymbol{X}})\leq\frac{1}{2}\left[\sqrt{\frac{\pi}{\tau}}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha)}\alpha\right]^{|\Omega|} \tag{15}$$

which could be compared to (12) of [13], where an approximate Chernoff bound is presented. Although the approximation in [13] which is given in terms of a truncated infinite summation works well for a large range of practical channel parameters, (15) provides a simpler result.

IV. NUMERICAL RESULTS

In this section, we will first compare the derived PEP bound with the exact PEP. Then, as an example, we will consider a convolutionally coded system and will use the derived PEP expression to compute upper bounds on the BER performance.

In Fig. 1, we plot derived bounds on PEP given by (14) for an error event of length 3 using channel parameters $\alpha=2$ and $\alpha=10$. We also compute the corresponding exact PEPs given by (7) using numerical integration and provide them as a reference (illustrated by solid lines). It is observed that the derived bounds coincide with the exact PEPs for high signal-to-noise ratios (SNR). Although the tightness of bound for small SNR values is low (i.e. the overlapping with the exact expression occurs asymptomatically), derived bounds capture well the behavior for a large range of SNR values. The PEP for the negative exponential channel (based on (11)), which can be considered as a limiting distribution for the *K* distribution is also included in Fig. 1. We observed that the PEP results over the *K* channel with $\alpha=40$ (not shown in the figure) lies very close to that for the negative exponential distribution. This is an expected result since the *K* distribution reduces to the negative exponential for the limiting case of $\alpha \to \infty$.

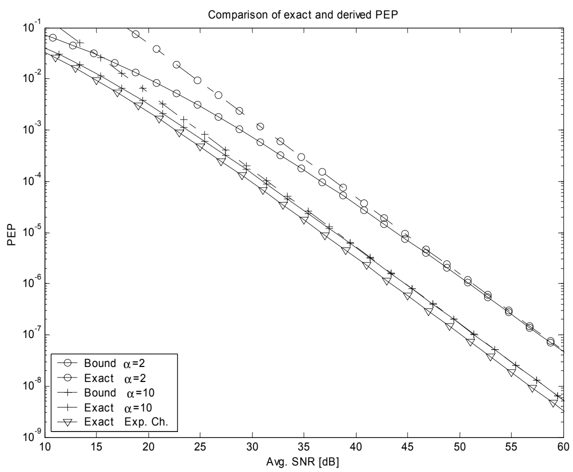

Fig. 1. Comparison of exact and derived PEPs for α=2 and α=10 (solid: exact, dashdot: derived bound)

It is obvious that the PEP is not the main issue in the performance evaluation of a coded communication system. One needs to consider bit or symbol error rate as an ultimate measure. A union bound on the average BER can be found as [15]

$$P_b \le \frac{1}{n}\sum_{\boldsymbol{X}} P(\boldsymbol{X}) \sum_{\boldsymbol{X}\neq\hat{\boldsymbol{X}}} q\left(\boldsymbol{X},\hat{\boldsymbol{X}}\right)P\left(\boldsymbol{X},\hat{\boldsymbol{X}}\right), \qquad (16)$$

where $P(\boldsymbol{X})$ is the probability that the sequence $\boldsymbol{X}$ is transmitted, $q\left(\boldsymbol{X},\hat{\boldsymbol{X}}\right)$ is the number of information bit errors in choosing another coded sequence $\hat{\boldsymbol{X}}$ instead of $\boldsymbol{X}$ and *n* is the number of information bits per transmission. Using transfer function bounding technique combined with the alternative form for the Gaussian-*Q* function, an efficient method for the computation of (16) is given as [12]

$$P_b \le \frac{1}{n}\sum_{\boldsymbol{X}} P(\boldsymbol{X})\frac{1}{\pi}\int_0^{\pi/2}\left[\frac{\partial}{\partial N}T(D(\theta),N)\bigg|_{N=1}\right]d\theta . \qquad (17)$$

For uniform error probability codes, a symmetry property exists, eliminating the need for averaging over all possible transmitted sequences. In this case, (17) simplifies to

$$P_b \le \frac{1}{\pi}\int_0^{\pi/2}\left[\frac{1}{n}\frac{\partial}{\partial N}T(D(\theta),N)\bigg|_{N=1}\right]d\theta . \qquad (18)$$

For the *K* channel, $D(\theta)$ is defined based on the derived PEP expression, i.e.

$$D(\theta) = \sqrt{\frac{\pi\sin^2\theta}{\tau}}\,\frac{\Gamma(\alpha-1)}{\Gamma(\alpha)}\alpha . \qquad (19)$$

In the case of exact PEP, $D(\theta)$ is given as

$$D(\theta) = \frac{2}{\Gamma(\alpha)}\alpha^{\frac{\alpha+1}{2}}\int_0^{\infty}\exp\left(-\frac{\tau}{4}\frac{I^2}{\sin^2\theta}\right)I^{\frac{\alpha-1}{2}}K_{\alpha-1}\left(2\sqrt{\alpha I}\right)dI . \qquad (20)$$

For the special case of $\alpha \to \infty$, exact PEP has a closed form solution as seen from (11) and $D(\theta)$ is given as

$$D(\theta) = 2\sqrt{\frac{\pi\sin^2\theta}{\tau}}\exp\left(\frac{\sin^2\theta}{\tau}\right)Q\left(\sqrt{\frac{2\sin^2\theta}{\tau}}\right) \qquad (21)$$

where we consider unity mean, i.e. $\mu = 1$, for simplicity.

As an example for demonstration of BER results, we consider a convolutionally coded system. The convolutional code under investigation [15] is illustrated in Fig. 2. It has a code rate of 1/3 and constraint length of 3.

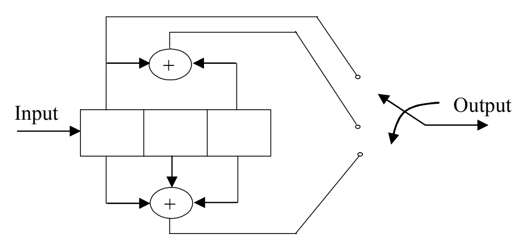

Fig. 2. Rate=1/3 convolutional encoder with constraint length 3 [15].

The average BER results are computed based on (18) in conjunction with (19) as well as with (20) to allow comparison with the true upper bound. Both of them are illustrated in the Fig. 3 for the *K* channel with parameters $\alpha = 2$ and $\alpha = 10$. As the limiting case, performance of exponential channel is also included for comparison purposes. For the all considered cases, upper bounds on BER based on the derived PEP are in good agreement with the true upper bound. Although there is some discrepancy in the lower SNR region, it provides good agreement as SNR increases. Monte-Carlo simulation results are furthermore included as a reference. Due to the long simulation time involved, we are able to give simulation results only up to BER=10^{-6}. Simulation results are observed to be located slightly lower than the true upper bound and demonstrate an excellent agreement with the analytical results. Considering BER=10^{-9} is a practical performance target for a FSO link, our analytical results can serve as a simple and reliable method to estimate BER performance without resorting to lengthy simulations.

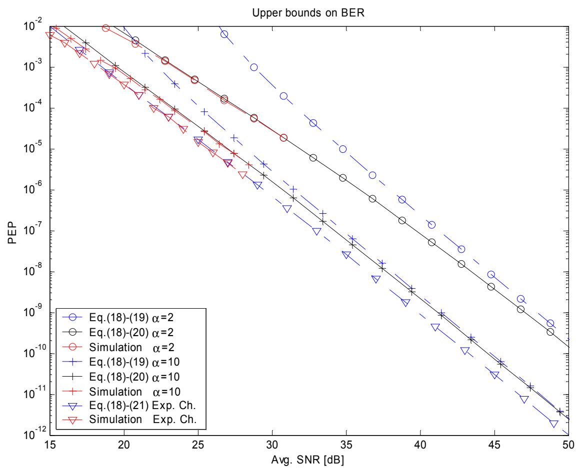

Fig. 3. Upper bounds on BER for the *K* channel (dashed dot blue: Eq. (18)-(19)/(21), solid black: Eq. (18)-(20), dashed red: Simulation)

V. CONCLUSIONS

In this paper, we derive error performance bounds for coded FSO communication systems operating over atmospheric turbulence channels, which are modeled as *K* distribution. Unlike the classically used log-normal assumption, this channel model describes strong turbulence conditions. We derive an upper bound on the PEP for the *K* channel in closed form and then apply the transfer function technique in conjunction with the derived PEP bound to obtain upper bounds on the BER performance. Simulation results are also included to confirm the analytical results.

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