Bandwidth Efficient Low Density Parity Check Coding using Multi Level Coding and Iterative Multi Stage Decoding

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Abstract: *The design of bandwidth efficient codes using low density parity check codes (LDPC) is studied. Bit interleaved coded modulation using LDPC codes and multilevel coding using LDPC codes are considered. An iterative decoding strategy which includes soft-decision feedback between the lower and higher levels in a multilevel coding scheme is proposed. Some practical considerations in designing LDPC codes with multilevel coding are addressed and, finally, the bit error rate performance and decoding complexity of the proposed coding techniques are compared with those of turbo trellis coded modulation techniques.*

Keywords: Multilevel coding, iterative demodulation, LDPC codes

1. Introduction

Since the introduction of trellis coded modulation (TCM) by Ungerböeck and multilevel coded (MLC) modulation by Imai and Hirakawa, bandwidth efficient coding has been an area of active research. Soon after the discovery of turbo codes, novel approaches have been undertaken to construct bandwidth efficient Turbo coding schemes which has resulted in Turbo trellis coded modulation (Turbo-TCM) [1] and parallel concatenated trellis coded modulation (PC-TCM) [2]. Multilevel coding schemes with Turbo codes as component codes have also been studied (see [3] for references). All these codes have performance remarkably close to the capacity, but the complexity is also significantly high.

Recently, there has been revised enthusiasm in low density parity check codes (LDPC codes or Gallager codes) which was first proposed by Gallager in 1962 [4]. Various investigations have shown that LDPC codes perform close to, if not as good as, Turbo codes, yet with possibly lesser complexity. A natural desire immediately arises to improve its bandwidth efficiency, which is the focus of this paper. We consider bit interleaved coded modulation (BICM) and a multilevel coded structure employing LDPC codes as component codes and compare their performance to turbo TCM. Further, we introduce an iterative multi-stage decoding algorithm which proves to be efficient for multilevel coding schemes. Simulation results show that MLC with LDPC codes performs close to that of turbo TCM, but with lesser complexity.

2. Background

2.1. LDPC Codes

A low density parity check (LDPC) code is a linear block code specified by a very sparse parity check matrix. The parity check matrix $\boldsymbol{H}$ of a (N, K, t) LDPC code of rate $R = K/N$ is a $N - K \times N$ matrix, which has t ones in each column and $j > t$ ones in each row. Apart from these constraints, the ones are placed at random in the parity check matrix. However, in practice, in order to avoid low weight code words, any two columns in the H matrix are not allowed to overlap in more than one non-zero bit position and t is ≥ 3. When the number of ones in every column is the same, the code is known as a regular LDPC code. When all the columns do not have same weight, the LDPC code is called irregular and, such codes are reported to outperform the regular LDPC codes. The decoding algorithm for LDPC codes is based on a belief propagation decoder and is explained in detail in [5].

2.2. Multilevel Coding

A multilevel code is a nested or leveled partitioning of signal constellations. An L-level partition is a partitioning chain, $\Gamma_0, \Gamma_1, \cdots, \Gamma_L$, which can be viewed as a rooted tree where the root, Γ_0, is the constellation itself [3]. In each level of partitioning, Γ_i/Γ_{i+1}, all points of the set are partitioned into disjoint subsets and an address bit x_i is used to pick one of the subsets. A component code $\mathcal{C}_i$ of appropriate length and rate is used to protect x_i and the decoder associated with the component code is $\mathcal{D}_i$. The design of the code rates is the key in a MLC scheme and is done based on the chain rule of mutual information

$$I(Y; X_0, X_1, \cdots, X_{L-1}) = I(Y; X_0) + I(Y; X_1|X_0) + \cdots + I(Y; X_{L-1}|X_0, \cdots, X_{L-2}) \quad (1)$$

where Y is the received signal. The i_{th} term of the right hand side corresponds to an equivalent channel of level i, over which the address bit x_i is transmitted, given that $x_0, x_1, \cdots, x_{i-1}$ are known. Wachsmann *et al* [3] showed that the overall capacity $\mathbf{C}$ can be achieved with a MLC scheme if and only if the individual rates R_i are chosen to be equal to the capacities of the equivalent channels, i.e., $R_i = \mathbf{C}_i$. Multilevel codes can be decoded using a suboptimal multi-stage decoding (MSD) algorithm which provides a good trade-off between the error performance and the decoding complexity. The MSD procedure starts by decoding the lowest level, and in each subsequent level of decoding, hard decisions from the previous (lower) level(s) are used to improve the performance at the higher levels.

3. Bandwidth Efficient LDPC Codes

In this section, we discuss a few techniques to construct bandwidth efficient LDPC by efficiently combining coding and modulation. For ease of exposition, we discuss an example - the design of a 2 bits/s/Hz coding scheme using 8 PSK signal constellation with code length N symbols (equivalently $3N$ bits).

3.1. Bit Interleaved Coded Modulation

Motivated by Caire *et al*'s [6] result that bit interleaved coded modulation with Gray labeling of signal points can perform very close to capacity limits, the first approach was to use a $(3N, 2N, 3)$ LDPC code. Unlike in the case of convolutional codes, an interleaver is unnecessary in this case, since the LDPC code inherently makes the adjacent coded bits independent. Therefore, groups of three bits $(3i, 3i+1, 3i+2)$ for $i = 0, 1, 2 \ldots, N$, were mapped to a symbol from the 8-PSK signal using Gray mapping. The decoder performs iterative demodulation and decoding similar to [7].

3.2. Multi-level Coded Modulation using LDPC codes

Another approach to constructing bandwidth efficient LDPC codes is by using LDPC codes as component codes in a multilevel coding scheme. The rates of the codes are derived from the capacities of the equivalent channels and natural mapping is used. The ideal rates for achieving an overall rate of 2 b/s/Hz with 8PSK modulation is [3] $R_0/R_1/R_2 = \mathbf{C}_0/\mathbf{C}_1/\mathbf{C}_2 = 0.2/0.81/0.99$

Although it was shown in [3] that if $R_i = \mathbf{C}_i$, suboptimal MSD suffices to achieve capacity, this is true only if the codes at each of the levels achieves capacity. For finite lengths, practical codes do not achieve capacity and, hence, MSD is not optimal and, hence, the performance of MSD can be significantly worse than that of the true ML decoder. Clearly, maximum-likelihood decoding would be impractical and, hence, we propose an iterative multi-stage decoder as an approximation to the true ML decoder.

3.2..1 Iterative Multi-stage Decoding (MSD with Feedback)

In conventional MSD, hard decisions are passed from the lower levels to the upper levels only. That is $\mathcal{D}_0 \to \mathcal{D}_1 \to \cdots \to \mathcal{D}_{L-1}$. There is no feedback to decoder $\mathcal{D}_0$ nor refinement of the estimates of $\mathcal{D}_0$. Consequently, any error in any stage of the decoding process is irrevocable and, hence, results in a frame error. Since practical codes of finite length always have non-zero error rates, the overall frame error rate is greater than that of the highest of the codes in the individual levels. Therefore, we consider feedback from higher levels to lower levels. It can be seen that feedback from decoder $\mathcal{D}_1$ could help the performance of decoder $\mathcal{D}_0$ for an 8-PSK constellation with natural mapping. Although feedback from $\mathcal{D}_1$ to $\mathcal{D}_0$ does not change the minimum distance for the address bit x_0, it does enlarge the decision region and, hence, $I(Y; x_0) \leq I(Y; x_0|x_1)$.

In order to explain the iterative multi-stage decoding process, let $(x_{0i}, x_{1i}, \ldots, x_{(L-1)i})$ denote the coded bits in the level 0,1, ..., $L-1$ that correspond to the ith symbol. The decoder comprises P stages of soft output feedback between the decoders $\mathcal{D}_l$, after Q iterations of LDPC decoding at each level l. The basic idea is to use the soft output produced by the component decoders $\mathcal{D}_1, \ldots, \mathcal{D}_{l-1}, \mathcal{D}_{l+1}, \ldots, \mathcal{D}_L$ after Q iterations as *a priori* information in the soft output demodulator that produces estimates of x_{li} based on the received signal. For this reason, we refer to the soft output feedback as iterative demodulation. At level l if the LDPC code is a (N, K_l) code, with parity check matrix H^l, then let R^l_j and C^l_i be two sets such that $R^l_j = \{\ i \mid H^l_{j,i} = 1\}$ and $C^l_i = \{\ j \mid H^l_{j,i} = 1\}$. The decoding algorithm for a 2^L-ary signal constellation with P stages of iterative demodulation and Q iterations at each level within one stage is then given in Table 1, where the function $\psi(x)$ refers to $\psi(x) = \log(\tanh(x/2))$

3.3. Practical Issues

The ideal rates for a 8-PSK modulation with natural mapping for the the three different levels is 0.2 / 0.81/ 0.99. For small block lengths (a few thousand bits), the construction of an LDPC code with rate 0.99 with $t = 3$ is practically impossible. Therefore, for short block lengths, a lower rate code has to be used for the highest level. This means using a higher rate code in the lower levels and, hence, the design rules cannot be exactly matched always. Fur-

Table 1: Decoding Algorithm

Pre-initialization: For $n = 0, 1, \cdots, N-1$
$\forall i, m = 1, 2, \ldots, 2^L, P_{ch}(s_{mi}) = A_i \cdot e^{-\frac{(r_i - s_{mi})^2}{2\sigma^2}}$ A_n is a normalization factor, s.t.: $\sum_{k=0}^{M} \Pr(s_{mi}) = 1$ $L_{ch}(x_{li}) = \log \frac{\sum_{s_{mi}:x_{li}=1} P_{ch}(s_{mi})}{\sum_{s_{mi}:x_{li}=0} P_{ch}(s_{mi})} \quad \forall l$
For $p = 1, 2, \ldots, P$**, for** $\mathcal{D}_l$**:** $l = 0, 1, \cdots, L-1$
Initialization: For $i = 0, 1, \cdots, N-1, \quad \forall j \in C_i^l$, $L_{ext}^{(p-1)}(x_{li}) = \log \frac{\sum_{s_{mi}:x_{li}=1} P_{ch}(s_{mi}) \prod_{k \neq l} P_j^{(Q,p-1)}(x_{ki})}{\sum_{s_{mi}:x_{li}=0} P_{ch}(s_{mi}) \prod_{k \neq l} P_j^{(Q,p-1)}(x_{ki})}$ where $P_j^{(Q,p-1)}(x_{li} = 0) = \frac{1}{1+e^{(L_j^{(Q,p-1)}(x_{li}) - L_{ch}(x_{li}))}}$ $L_j^{(0,p)}(x_{li}) = L_{ch}(x_{li}) + L_j^{(Q,p-1)}(x_{li}) + L_{ext}^{(p-1)}(x_{li})$
q**th Iteration in** $\mathcal{D}_l$**:** $q = 1, 2, \cdots, Q$
For $j = 0, 1, \cdots, N-K-1$ $M1 = \sum_{i \in R_j^l} \psi(L_j^{(q-1,p)}(x_{li}))$ $S1 = \prod_{i \in R_j^l} sign(L_j^{(q-1,p)}(x_{li}))$ $\forall\ i \in R_j^l,\ L_{ej}^{(q,p)}(x_{li}) = S1 \times sign(L_j^{(q-1,p)}(x_{li}))$ $\times \psi(M1 - \psi(L_j^{(q-1,p)}(x_{li}))$
Extrinsic info: For $i = 0, 1, \cdots, N-1,, \forall j \in C_i^l$, $L_j^{(q,p)}(x_{li}) = L_{ch}(x_{li}) + \sum_{m \in C_i^l \setminus \{j\}} L_{em}^{(q,p)}(x_{li})$
Soft output: For $i = 0, 1, \cdots, N-1$, $L(x_{li}) = L_c(x_{li}) + \sum_{j \in C_i^l} L_{ej}^{(q,p)}(x_{li})$

ther, When the block lengths are small (about 1000 bits), it is quite difficult to construct good LDPC codes of very high rate. For example, for a block length of $N = 1024$, it is quite difficult to construct good LDPC codes of rate > 0.95. Therefore, we have used a 4 error correcting Bose Chaudhuri and Hocquenghem (BCH) code for the highest level for block size of 1024 bits instead of LDPC codes.

At the lowest level, the effective signal to noise ratio as seen by the code is very low and, hence, codes that perform well at such low SNRs should be used. Although it is known that if $t \geq 3$, the resulting LDPC code has good distance spectrum, at low SNRs, the performance is dominated by the convergence of the belief propagation decoding algorithm. Consequently, codes that help convergence should be used. The convergence of the decoding algorithm depends on the number (and girth) of cycles in the graph of the LDPC code. Since an average column weight corresponds to lesser probability of having several cycles, the performance of LDPC codes with low column weight was considered. Experimentally, it was found that an average column weight of approximately 2.3 performed best for bit error rates in the range of 10^{-5} and, therefore, for the lowest level LDPC codes with $t = 2.3$ were used.

Low density parity check codes with very low column weight have codewords with small Hamming weight and, hence, the bit error rate (BER) performance of such codes exhibits a BER floor. This floor is due to a few codewords with small Hamming weight that cause errors in a few fixed bit positions, which can be determined. If dummy bits are used in these bit positions, then the error floor can be reduced [8]. We identified a set of few error prone bit positions through simulations and used dummy bits in these positions. This set of bit positions has very few elements and, hence, this does not reduce the bandwidth efficiency significantly.

4. Decoding Complexity

The total decoding complexity is the sum of the decoding complexities in each level and the complexity for iterative demodulation. We assume that all operations of the type $\log(e^{a_1} + e^{a_2})$ is implemented using $\max(a_1, a_2) + \log(1 + e^{-|a_1 - a_2|})$, where the second term is implemented using a look-up table. For decoding of LDPC codes, the function $\psi(a)$ is implemented using a look up table. For an (N, K) binary LDPC code with average column weight t, the row operations require $4Nt$ look ups and $4Nt$ adds/subtracts and $4Nt$ multiplies by ± 1 (sign operations). The column operations require approximately $2Nt$ adds/subtracts. Before we compute the complexity for iterative demodulation (soft output feedback), we note that iterative feedback between all the levels is not required always. For 8PSK modulation considered here, iterative exchange between level 1 and level 2 was sufficient. Hence, we compute complexity of iterative exchange between level 1 and level 2 only. Iterative demodulation requires $16N$ additions, $8N$ max operations, $8N$ lookups per update. To get an approximate idea of the complexity, we treat all the operations as one flop and ignore the complexity in sign operations (multiply by ± 1). Assuming that the number of iterations at the lth level is Q_l (rather than being fixed at Q), the total number of operations per information bit for 2 b/s/Hz 8PSK scheme is given by $\frac{2}{N}(\sum_l 10Nt_l Q_l + 2 \times 32N)P$.

In all the simulations, iterative feedback is used only between levels 1 and 2 and the third level is decoded once using soft decision feedback from levels 1 and 2. For the levels 1 and 2, Q_1 and Q_2 was set at 3 and P was set at 30 and, hence, the maximum number of iterations is 90. However, for medium to high SNRs, the average number of stages required was small (about 7).

The complexity of turbo TCM scheme (and parallel concatenated TCM) with 2^M states and 2^K transitions per state for a 2^L ary signal constellation can be shown to be approximately $\frac{13 \times 2^K 2^M + 2^K 2^L + 2^K + (K-1)}{K}$ The complexity of the turbo TCM scheme of Robertson and Wörz can be seen to be approximately twice as complex as the present scheme for 8 state codes both based on average number of iterations and based

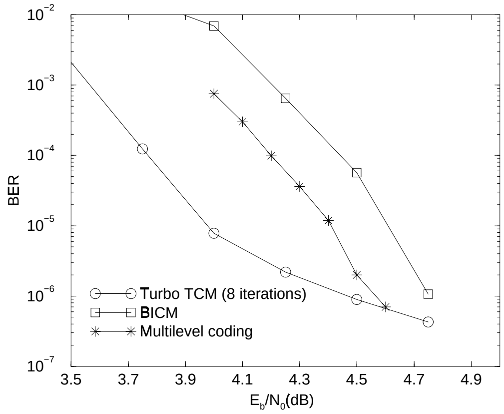

Figure 1: Performance of BICM, multilevel modulation with LDPC codes and Turbo TCM; $N = 1024$ symbols

on worst case. The complexity of parallel concatenated trellis coded modulation [2] is approximately twice as that of turbo TCM. Hence, the proposed method has significant complexity advantage.

5. Simulation Results and Conclusion

Fig 1 shows the performance of multilevel coding with iterative multistage decoding on an AWGN channel with 8PSK modulation. The performance of Robertson and Wörz's turbo TCM and BICM with iterative demodulation is also shown. It can be seen that multilevel coding performs within 0.4 dB of TTCM at high BERs and outperforms turbo TCM for BERs $< 10^{-6}$. The complexity of the proposed scheme is significantly lesser than that of the turbo TCM scheme. Although not compared here, the performance of parallel concatenated trellis coded modulation, proposed by Benedetto *et al* performs slightly better than the multilevel coded scheme proposed here. However, the complexity of the proposed scheme is significantly lower. In order to show the performance improvement due to iterative multistage decoding, the performance of multilevel coding with $P = 1, Q_1 = Q_2 = Q_3 = 100$ is compared to that with $P = 30, Q_1 = Q_2 = 30, Q_3 = 20$ in Fig. 2 for $N = 3000$ symbols. The first case corresponds to conventional multistage decoding (however, with soft decision feedback). The code rates used were 0.2 / 0.82266 / 0.966 and the average column weights were $t_1 = 2.3$, $t_2 = 3.1$ and $t_3 = 2.2$. It can be seen that iterative multistage decoding offers improved performance. It can also be seen that the performance of the proposed scheme is within 1 dB of the constrained capacity at BER of 10^{-5} which is 2.9 dB for 8PSK modulation. Although not shown there, the performance of the proposed scheme for $N = 2000$ was identical that of using 16-state turbo codes as component codes [3].

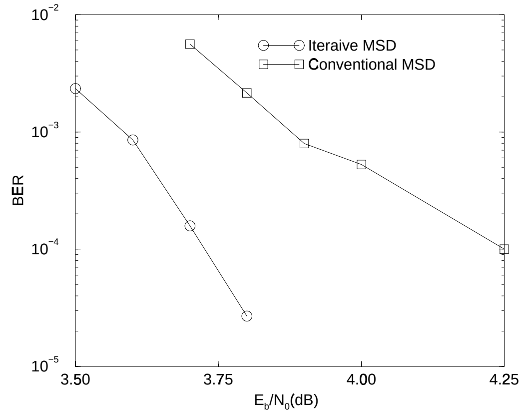

Figure 2: Performance of iterative and conventional MSD; $N = 3000$ symbols

We have compared different design methodologies for constructing bandwidth efficient low density parity check codes. The use of multilevel coding with iterative multistage decoding is a promising choice due to its good performance and low decoding complexity.

REFERENCES

[1] P. Roberston and T. worz, "A novel bandwidth efficient coding scheme employing turbo codes," in *IEEE Int. Conf. on Commun.*, pp. 962–967, 1996.

[2] S. Benedetto, D. Divsalar, G. Montorsi, and F. Pollara, "Parallel concatenated trellis coded modulation," in *IEEE Int. Conf. on Commun.*, pp. 974–978, 1996.

[3] U. Wachsmann, R.F.H.Fischer, and J.B.Huber, "Multilevel codes : Theoretical concepts and practical design rules," *IEEE Trans. Info. Theory*, vol. 45, pp. 1361–1391, July 1999.

[4] R. Gallager, "Low-density parity-check codes," *IRE Transactions on Information Theory*, January 1962.

[5] J. Hagenauer, E. Offer, C. Meason, and M. Morz, "Decoding and equalization with analog nonlinear networks," *European Trans. Telecom*, Oct. 199.

[6] G. Caire, G. Taricco, and E. Biglieri, "Bit-interleaved coded modulation," *IEEE Trans. Info. Theory*, pp. 927–947, May 1998.

[7] X. Li and J. Ritcey, "Trellis-coded modulation with bit interleaving and iterative decoding," *IEEE J. Selec. Areas Commun.*, pp. 715–724, Apr. 1999.

[8] M. Öberg and P. Siegel, "Lowering the error floor of turbo codes," in *Intl. Symp. on Turbo Codes and Related Topics*, pp. 204–207, 1997.